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The boundary layer problem: A fourth-order adaptive collocation approach

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ABSTRACT

A finite element approach, based on the cubic *B*-spline collocation, is presented for the numerical solution of a class of singularly perturbed two-point boundary value problems that possesses a boundary layer at one or two end points. Due to the existence of a layer, the problem is handled using an adaptive spline collocation approach constructed over a non-uniform Shishkin-like mesh, defined via a carefully selected generating function. To tackle the case of nonlinearity, if it exists, an iterative scheme arising from Newton's method is employed.

The rate of convergence is verified to be of fourth-order and is calculated using the double-mesh principle. The efficiency and applicability of the method are demonstrated by applying it to a number of linear and nonlinear examples. The numerical solutions are compared with both analytical and other existing numerical solutions in the literature. The numerical results confirm that this method is superior when contrasted with other accessible approaches and yields more accurate solutions.

1. Introduction

In this section, a fourth-order finite-element solution approach using cubic *B*-splines is presented for the following singularly perturbed boundary value problems:

$$-\epsilon y'' + p(x)y' + q(x)y = f(x, y), \tag{1.1}$$

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta, \tag{1.2}$$

where $a \leq x \leq b$, α, β are constants, ϵ is a very small positive perturbation parameter, and $p(x)$ and $q(x)$ are sufficiently smooth real-valued functions. Normally, additional assumptions such as boundedness are imposed on the coefficients (see [1]) in order to guarantee the existence of a unique solution and to have the layer in the neighborhood of one endpoint.

This problem arises in transport phenomena in chemistry and biology (see [2,3] and the references therein) and has been studied extensively in the literature, for instance [3–5]. The problem is attractive and of special interest due to the existence of a thin boundary layer over which the solution varies quite rapidly, while away from the layer the solution behaves regularly and varies slowly, which complicates the numerical treatment of the problem and causes computational difficulties.

In recent years, there was a broad attention and increased focus by several authors on the numerical solution of singularly perturbed problems. A number of approaches were manipulated to tackle the boundary layer problem (1.1)–(1.2) and other similar or special versions of the problem. Lin et al. [3] used the *B*-spline collocation method to develop a numerical method for solving the corresponding homogeneous singularly perturbed boundary value problem of Eq. (1.1) that lead

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to a tridiagonal linear system. Wong and Yang [4] discussed the corresponding homogeneous case of (1.1) and used the method of successive approximation to show that under certain smoothness conditions on $p(x)$ and $q(x)$, an asymptotic solution is constructed, which holds uniformly for all x inside the given domain $a \le x \le b$. In a second paper [5], the same authors obtained an asymptotic solution for the same problem with new conditions imposed on the coefficients. Reddy and Chakravarthy [1] proposed a numerical patching method for the singularly perturbed two-point boundary value problem (1.1) where f is a function of x, using cubic splines. The method is distinguished by the following fact. The original problem is divided into two problems, namely, inner and outer region problems; the terminal boundary condition is obtained from the solution of the reduced problem. Using general stretching transformation, a modified inner region problem is constructed then both the inner region problem and the outer region problem are solved as two-point boundary value problems by employing cubic splines.

The solution for such problems exhibits a multiscale character, that is, the solution varies rapidly within the layer and varies slowly outside it. Piecewise-uniform Shishkin meshes (see [6,7]) played a major role to tackle these problems. These piecewise-uniform grids played a prominent role in numerical methods and presented an efficient strategy to handle the effects of the boundary layer. Lately, there has been substantial interest in implementing and analyzing methods based on these meshes and other closely related ones. Consequently the numerical method must be tailored very carefully to undertake the layer and must be robust in the sense that the error in the approximation must not deteriorate as the singular perturbation parameter ϵ approaches zero. The piecewise-uniform Shishkin meshes are not usually optimal but fairly attractive due to its simplicity and suitability in handling a wide range of singularly perturbed problems. The difficulty in the implementation of the Shishkin meshes is that one should have a priori knowledge about the nature and thickness of the boundary layer.

In this paper, we present an adaptive grid technique [8], based on the B-spline collocation [9] on non-uniform Shishkin-like meshes for the solution of the boundary layer problem. The cubic B-spline finite element method (see [10–16,3,7,1]) is often used for solving nonlinear problems [17] that arise in engineering applications. The approach necessitates the redistribution of the nodes in order to have more points placed in regions of large variation of the solution, that is close to layers; the mesh is finer near the boundary layer but coarser otherwise.

The spline collocation method has been integrated with the adaptive technique to solve boundary-value problems on non-uniform meshes by mapping uniform node points to non-uniform ones such that the errors are reduced [10]. For instance, Khuri and Sayfy [11] used a similar finite element approach for the numerical solution of Troesch's problem. The method is used on both a uniform mesh and a piecewise-uniform Shishkin mesh, depending on the magnitude of the eigenvalues. This is due to the existence of a boundary layer at the right endpoint of the domain for relatively large eigenvalues. The problem is also solved using an adaptive spline collocation approach over a non-uniform mesh via exploiting an iterative scheme arising from Newton's method.

The numerical solutions are compared with both the analytical solutions and the other existing numerical solutions in the literature. The convergence analysis is discussed and it is shown that the method has a fourth-order rate of convergence using the double-mesh principle. To demonstrate the performance and efficiency of the method a number of examples are considered. It is observed that the results obtained show the efficiency and superiority of this method compared to other available solutions.

The balance of this paper is organized as follows. In Section 2, the cubic B-spline finite element method over a uniform mesh and the adaptive cubic B-spline over a non-uniform mesh approaches for the numerical solution of the boundary layer problem are described. In Section 3, the technique is implemented for a number of special cases of the problem using the adaptive technique on non-uniform and Shishkin-like meshes. The results are compared with the exact solutions and some existing numerical solutions. Finally, in Section 4 a conclusion is given that briefly summarizes the results of the paper's content.

2. Numerical approach

In this section, the two numerical techniques are presented for solving the boundary layer problem (1.1)–(1.2) on the interval $[a, b]$. The approximate solution will be constructed over the node points x_i, where

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b.$$

2.1. The B-spline collocation method over a uniform mesh

First, we will seek a finite-element solution over a uniform mesh. For this case we have $x_i = a + ih,\ i = 0, 1, 2, \ldots, N$ where $h = \frac{b-a}{N}$. Then let $\psi(x)$ be a shape function that satisfies the two boundary conditions (1.2) and is expressed as a linear combination of $N + 3$ spline functions given by

$$\psi(x) = \sum_{i=-3}^{N-1} a_i \psi_i(x). \tag{2.3}$$

Table 1
Values of ψ_i, ψ_i' and ψ_i'' at the node points.

B-spline	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
ψ_i	0	1	4	1	0
ψ_i'	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
ψ_i''	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

The a_i's are the unknown real coefficients and the $\psi_i(x)$ are the cubic B-spline functions defined as follows:

$$\psi_i(x) = \frac{1}{h^3}\begin{cases} (x-x_i)^3, & [x_i, x_{i+1}] \\ h^3 + 3h^2(x-x_{i+1}) + 3h(x-x_{i+1})^2 - 3(x-x_{i+1})^3, & [x_{i+1}, x_{i+2}] \\ h^3 + 3h^2(x_{i+3}-x) + 3h(x_{i+3}-x)^2 - 3(x_{i+3}-x)^3, & [x_{i+2}, x_{i+3}] \\ (x_{i+4}-x)^3, & [x_{i+3}, x_{i+4}] \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where $h = x_{i+1} - x_i$. From (2.4), the values of ψ_i, ψ_i' and ψ_i'' at the node points $x_i = a + ih$ are given according to the following table.

We assume that the solution of problem (1.1) can be approximated by (2.3) where the coefficients $\{a_i\}$ are to be determined. Substituting the solution given in (2.3) into the differential equation (1.1) yields the following nonlinear system of equations.

$$-\epsilon\sum_{i=-3}^{N-1} a_i\psi_i''(x_j) + \sum_{i=-3}^{N-1} a_i p(x_j)\psi_i'(x_j) + \sum_{i=-3}^{N-1} a_i q(x_j)\psi_i(x_j) = f\left(\sum_{i=-3}^{N-1} a_i\psi_i(x_j)\right), \quad j = 0, 1, 2, \dots, N. \tag{2.5}$$

This system consists of $N+1$ equations in $N+3$ unknowns. The boundary conditions in (1.2) give the following two equations.

For $x = a$ we have

$$y(a) = \sum_{i=-3}^{N-1} a_i\psi_i(x_0) = \alpha. \tag{2.6}$$

For $x = b$ we have

$$y(b) = \sum_{i=-3}^{N-1} a_i\psi_i(x_n) = \beta. \tag{2.7}$$

The values of $\psi_i(x_j)$, $\psi_i'(x_j)$ and $\psi_i''(x_j)$ at the nodal points x_j, $j = 0, 1, \dots, N$ are determined from Table 1.

Therefore, the $(N+3) \times (N+3)$ system of equations in (2.5)–(2.7) can be written in matrix form as follows:

$$\mathbf{Cd} = \mathbf{f}, \tag{2.8}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ r_0 & s_0 & p_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & s_1 & p_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & r_N & s_N & p_N \\ 0 & 0 & 0 & \cdots & 1 & 4 & 1 \end{bmatrix}.$$

We have

$$r_j = -\frac{6\epsilon}{h^2} - \frac{3p_j}{h} + q_j, \qquad s_j = \frac{12\epsilon}{h^2} + 4q_j, \qquad p_j = -\frac{6\epsilon}{h^2} + \frac{3p_j}{h} + q_j, \quad j = 0, 1, \dots, N,$$

given that

$$p_j = p_{(x_j)}, \qquad q_j = q(x_j) \quad \text{where } x_j = a + jh,$$

$$\mathbf{f}=\begin{bmatrix}\alpha\\ f\,(a_{-3}+4a_{-2}+a_{-1})\\ f\,(a_{-2}+4a_{-1}+a_0)\\ f\,(a_{-1}+4a_0+a_1)\\ \cdot\\ \cdot\\ \cdot\\ f\,(a_{N-4}+4a_{N-3}+a_{N-2})\\ f\,(a_{N-3}+4a_{N-2}+a_{N-1})\\ \beta\end{bmatrix},$$

and

$$\mathbf{d}=\begin{bmatrix}a_{-1}, & a_0, & \cdots, & a_{N-3}\end{bmatrix}^T.$$

The nonlinear system of equations given in (2.8) is solved using the computer algebra system *Maple*. Due to the limitation of *Maple*, the nonlinear system cannot be solved for fine uniform meshes (that is, for high dimensional system). On the other hand, to solve boundary value problems we need finer meshes only within the domain that is contained with the boundary layer whose length is less than the thickness of the layer. For this reason we use instead an adaptive technique bases on non-uniform meshes.

2.2. Adaptive spline collocation over a non-uniform mesh

In this subsection, we present the adapted spline collocation technique, based on non-uniform nodes, for the numerical solution of the singularly perturbed boundary-value problem (1.1)–(1.2). It is worth mentioning that this adaptive method applies exclusively for linear problems and, in the case of a nonlinearity, an iteration scheme arising from Newton's method should be applied first in order to linearize the problem.

To implement the approach, we need to opt a strictly increasing bijective function that maps the uniform nodes $\{x_i\}$ to non-uniform meshes with nodes $\{w_i\}$ which are positioned appropriately in order to reduce the error. In an ideal situation, one would like to redistribute the nodes such that we have almost the same error at each step, that is, the error is uniformly distributed along the meshes (see [10]). To achieve this, we need to carefully select a tailored grading function that redistributes the nodes with more near the boundary layer, that is to say, we require a finer mesh in the boundary layer region and coarser mesh in the regular region. For our problem, we chose to define the grading function [8] $w := [a, b] \to [a, b]$ to be one of the following:

Type I. Grading functions:

$$w(x)=(b-a)\left[1-\frac{(1+k)^{\left(1-\frac{x-a}{b-a}\right)}-1}{k}\right]+a, \tag{2.9}$$

$$w(x)=(b-a)\left[\frac{(1+k)^{\frac{x-a}{b-a}}-1}{k}\right]+a. \tag{2.10}$$

The grading function $w(x)$ redistributes the nodes with more nodes near $x = a$ or near $x = b$ according to Eqs. (2.9) or (2.10), respectively.

Type II. Chebyshev–Gauss–Lobatto points (Lobatto points in short):

$$x_i=\frac{1}{2}\left(1-\cos\left(\frac{i-1}{N-1}\pi\right)\right). \tag{2.11}$$

To describe the method, for the purpose of constructing an approximate solution on [0, 1], we first consider a uniform node point $x_0 < x_1 < \cdots < x_N$, where $x_i = ih,\ i = 0, 1, 2, \ldots, N$; $h = 1/N$. Then the above grading functions are used to redistribute the nodes over the interval [0, 1]. More specifically, we have the following.

(a) To have more nodes near $x = 0$ we use the Type I grading function on the interval [0, 1]:

$$w_i=\frac{1+k-(1+k)^{1-x_i}}{k}.$$

For more nodes near $x = 1$ we use

$$w_i=\frac{(1+k)^{x_i}-1}{k}.$$

As k increases more nodes lie near the boundary layer.

(b) For a problem with layers at both ends, we use the Type II Chebyshev–Gauss–Lobatto function

$$z_i = \frac{1}{2}\left(1 - \cos(i\pi/N)\right), \quad i = 0, 1, \dots, N.$$

For a finer mesh near the boundaries we use the transformation

$$w_i = (1-\tau)\left(3z_i^2 - 2z_i^3\right) + \tau z_i,$$

where τ is the adjustment parameter. In case a redistribution is required on an interval $[a, b]$, we can apply a simple linear transformation in order to map the interval $[0, 1]$ onto $[a, b]$.

The mesh selection strategy is based on Shishkin-like meshes [7], for which the thickness of the boundary layer is approximated first by

$$\sigma = \min\left(\frac{1}{2}, \frac{\epsilon \ln N}{a^*}\right),$$

where a^* is a parameter independent of N and ϵ. Then the solution interval Ω is divided as the union of two subintervals $\Omega = \Omega_1 \cup \Omega_2$ where either

$$\Omega = [0, \sigma] \cup [\sigma, 1],$$

or

$$\Omega = [0, 1-\sigma] \cup [1-\sigma, 1],$$

depending on whether the location of the boundary layer is at $x = 0$ or $x = 1$, respectively. Note that the thickness of the boundary layer, namely σ, will serve as the transition parameter between the two inner and outer regions. The number of points N is also divided between each part, depending on the value of σ.

The adaptive technique is designed for linear problems. Thus, we will present its implementation for the numerical solution of the following linear boundary-value problem:

$$L[u(w)] \equiv r(w)u'' + p(w)u' + q(w)u = g(w), \tag{2.12}$$

where $w \in \Omega = (a, b)$, and the specified boundary conditions

$$u(a) = \gamma_0, \qquad u(b) = \gamma_1. \tag{2.13}$$

The solution $u(w)$ is approximated by $\Psi(w)$ which is a linear combination of spline functions given by

$$\Psi(w) = \sum_{i=-3}^{n-1} c_i \Psi_i(w), \tag{2.14}$$

where $\Psi_i(w)$ is the non-uniform spline function defined by

$$\psi_i(x) = \begin{cases} \dfrac{(x-w_i)^3}{w_{i,3}\, w_{i,2}\, w_{i,1}}, & w_i \le x \le w_{i+1} \\[2ex] \dfrac{x-w_i}{w_{i,3}}\left[\dfrac{(x-w_i)(w_{i+2}-x)}{w_{i,2}\, w_{i+1,1}} + \dfrac{(x-w_{i+1})(w_{i+3}-x)}{w_{i+1,2}\, w_{i+1,1}}\right] + \dfrac{(x-w_{i+1})^2(w_{i+4}-x)}{w_{i+1,3}\, w_{i+1,2}\, w_{i+1,1}}, & w_{i+1} \le x \le w_{i+2} \\[2ex] \dfrac{w_{i+4}-x}{w_{i+1,3}}\left[\dfrac{(x-w_{i+1})(w_{i+3}-x)}{w_{i+1,2}\, w_{i+2,1}} + \dfrac{(x-w_{i+2})(w_{i+4}-x)}{w_{i+2,2}\, w_{i+2,1}}\right] + \dfrac{(x-w_i)(w_{i+3}-x)^2}{w_{i,3}\, w_{i+1,2}\, w_{i+2,1}}, & w_{i+2} \le x \le w_{i+3} \\[2ex] \dfrac{(w_{i+4}-x)^3}{w_{i+1,3}\, w_{i+2,2}\, w_{i+3,1}}, & w_{i+3} \le x \le w_{i+4} \\ 0, & \text{otherwise,} \end{cases} \tag{2.15}$$

where $w_{i,j} = w_{i+j} - w_i$. The coefficients c_i given in the definition of $\Psi(w)$ in (2.14) are the solution of the following system of linear equations:

$$\begin{aligned} &\Psi(w_0) = \gamma_0 \\ &\Psi(w_N) = \gamma_1 \\ &L\left[\Psi(w_j)\right] = g(w_j) - \mathcal{L}_p\left[\Psi(w_j)\right], \quad j = 0, 1, \dots, N \end{aligned} \tag{2.16}$$

where $\mathcal{L}_p$ is the operator which gives optimum order [10] and is defined as follows:

$$\begin{aligned}
\mathcal{L}_p\left[\Psi(w_0)\right] &= \frac{r(w_0)h_0}{24}\left(5h_0-4h_1+h_2\right)\frac{(h_0+h_1)\Gamma\left[S''(w_1)\right]-h_0\Gamma\left[S''(w_2)\right]}{h_1}\\
\mathcal{L}_p\left[\Psi(w_N)\right] &= \frac{r(w_N)}{24}h_{N-1}\left(5h_{N-1}-4h_{N-2}+h_{N-3}\right)\frac{(h_{N-1}+h_{N-2})\Gamma\left[S''(w_{N-1})\right]-h_{N-1}\Gamma\left[S''(w_{N-2})\right]}{h_{N-2}}\\
\mathcal{L}_p\left[\Psi(w_j)\right] &= \frac{r(w_j)}{12}h_jh_{j-1}\Gamma\left[S''(w_j)\right],\quad j=1,2,3,\ldots,N-1
\end{aligned}\tag{2.17}$$

where

$$\Gamma\left[S(w_j)\right]=\frac{2h_jS(w_{j-1})-2\left(h_{j-1}+h_j\right)S(w_j)+2h_{j-1}S(w_{j+1})}{h_{j-1}\left(h_{j-1}+h_j\right)h_j}.\tag{2.18}$$

For a uniform partition the above equations reduce to

$$\begin{aligned}
\mathcal{L}_p\left[\Psi(w_0)\right] &= \frac{r(w_0)h^2}{12}\left(2\Gamma[S''(w_1)]-\Gamma[S''(w_2)]\right)\\
\mathcal{L}_p\left[\Psi(w_N)\right] &= \frac{r(w_n)h^2}{12}\left(2\Gamma[S''(w_{n-1})]-\Gamma[S''(w_{n-2})]\right)\\
\mathcal{L}_p\left[\Psi(w_j)\right] &= \frac{r(w_j)h^2}{12}\Gamma[S''(w_j)],\quad j=1,2,3,\ldots,N-1
\end{aligned}\tag{2.19}$$

where for a uniform partition $\Gamma[S(w_j)]$ reduces to the three-point central difference approximation formula for the second derivative, namely,

$$\Gamma\left[S(w_j)\right]=\frac{S(w_{j-1})-2S(w_j)+S(w_{j+1})}{h^2}.\tag{2.20}$$

Next, we find the values of the spline functions together with their first and second derivatives at the nodes $w_{i+1}, w_{i+2}, w_{i+3}$. Upon denoting

$$\Psi_i^{(r)}=\left[\Psi_i^{(r)}\left(\Psi_{i+1}\right),\Psi_i^{(r)}\left(w_{i+2}\right),\Psi_i^{(r)}\left(w_{i+3}\right)\right],\tag{2.21}$$

for $r=0,1,2$, system (2.16) reduces to the following $N+3$ linear system:

$$\begin{aligned}
&\Psi_{-3,3}\,c_{-3}+\Psi_{-2,2}\,c_{-2}+\Psi_{-1,1}\,c_{-1}=\gamma_0,\\
&\Psi_{N-3,3}c_{N-3}+\Psi_{N-2,2}c_{N-2}+\Psi_{N-1,1}c_{N-1}=\gamma_1,\\
&r(w_j)\left\{\Psi''_{j-3,3}\,c_{j-3}+\Psi''_{j-2,2}\,c_{j-2}+\Psi''_{j-1,1}\,c_{j-1}\right\}+p(w_j)\left\{\Psi'_{j-3,3}\,c_{j-3}+\Psi'_{j-2,2}\,c_{j-2}+\Psi'_{j-1,1}\,c_{j-1}\right\}\\
&\quad+\left(q(w_j)I+\mathcal{L}_p\right)\left[\Psi_{j-3,3}\,c_{j-3}+\Psi_{j-2,2}\,c_{j-2}+\Psi_{j-1,1}\,c_{j-1}\right]=g(w_j),\quad\text{for } j=1,2,\ldots,N-1.
\end{aligned}\tag{2.22}$$

Here I is the identity operator.

The previous analysis is limited to linear problems. Assume there exists a nonlinear term in (2.12), so the problem reads as follows:

$$L\left[u(w)\right]\equiv r(w)u''+p(w)u'+q^*(w)u=g^*(w,u),\tag{2.23}$$

where $w\in\Omega=(a,b)$, and the same boundary conditions

$$u(a)=\gamma_0,\qquad u(b)=\gamma_1.\tag{2.24}$$

Then, to apply the adaptive technique we first apply the subsequent iteration scheme (2.25) arising from Newton's method, then solve the resulting linear equation using the adaptive technique iteratively:

$$\begin{aligned}
&r(w)u''_m+p(w)u'_m+q^*(w)u_m-\frac{\partial g^*(w,u_{m-1})}{\partial u}u_m=g^*(w,u_{m-1})-\frac{\partial g^*(w,u_{m-1})}{\partial u}u_{m-1}\\
&u(a)=\gamma_0,\qquad u(b)=\gamma_1.
\end{aligned}\tag{2.25}$$

In other words, we start with $u_0=u(a)$ and solve (2.25), for $m=1,2,3,4,\ldots,M$, using the previously described adaptive approach for linear problems, where $q(w)$ and $g(w)$ in Eq. (2.12) are given by

$$\begin{aligned}
q(w)&=q^*(w)-\frac{\partial g(w,u_{m-1})}{\partial u},\\
g(w)&=g^*(w,u_{m-1})-\frac{\partial g(w,u_{m-1})}{\partial u}\,u_{m-1}.
\end{aligned}\tag{2.26}$$

Here M is a positive integer such that $|u_m-u_{m-1}|$ is less than a given tolerance.

2.3. Discussion of convergence of method

In this subsection, we discuss the convergence of the method and show that the order of convergence is four.

For any given $w(x)$, let $u^3_{\Delta_w}$ denote the cubic spline space with respect to the partition Δ_w. In particular, let $u^3_{\Delta[1]}$ be the spline that satisfies

$$\begin{aligned} Lu^3_{\Delta[1]} &= g \quad \text{in } \Delta_w, \\ Bu^3_{\Delta[1]} &= \gamma \quad \text{on } T_{w\mathbf{B}}, \end{aligned} \tag{2.27}$$

and then u^3_Δ to be the spline which is forced to satisfy

$$\begin{aligned} Lu^3_{\Delta} &= g - \mathcal{L}_p u^3_{\Delta[1]} \quad \text{in } \Delta_w, \\ Bu^3_{\Delta} &= \gamma \quad \text{on } T_{w\mathbf{B}}. \end{aligned} \tag{2.28}$$

Here L is the operator given in Eq. (2.12), $\mathcal{L}_p$ defined in (2.17), while B is any specified mixed boundary conditions and for our case it is as given in (2.13). As for $T_{w\mathbf{B}}$, it is the set of boundary collocation points with respect to w. The next two theorems are taken from [10] (namely, Theorems 12, and 13, respectively).

Theorem 1 *([10]). If*

(A1) *the coefficients p and q, and the right-side g are $\mathbf{C}[\Omega]$,*
(A2) *the BVP $L[u] = g$, $Bu = 0$ has a unique solution,*
(A3) *the BVP $u'' = 0$, $Bu = 0$ has a unique solution,*
(A4) *$u \in \mathbf{C^6}[\overline{\Omega}]$, $w(x) : \overline{\Omega} \to \overline{\Omega}$ is a bijective map in $\mathbf{C}^3$, with $w'(x) > 0$, $\forall x \in \overline{\Omega}$, $w^{-1} \in \mathbf{C^1}[\overline{\Omega}]$,*

then $u^3_\Delta \in S^3_{\Delta_w}$ defined by (2.27) exists, is unique, and satisfies the global error estimates

$$\begin{aligned} \left\| \left(u - u^3_{\Delta[1]}\right)^{(k)} \right\|_\infty &= \mathcal{O}\left(h^2\right), \quad k = 0, 1, 2, \\ \left\| \left(u - u^3_{\Delta[1]}\right)^{(3)} \right\|_\infty &= \mathcal{O}(h), \end{aligned}$$

and the local error estimates

$$\left\| \left(u - u^3_{\Delta[1]}\right)^{(3)} (w_i) \right\|_\infty = \mathcal{O}\left(h^2\right), \quad i = 1, 2, \dots, N.$$

Define the Gaussian points $\delta_{ij} = x_i - \lambda_j h;\ j = 1, 2,\ i = 1, \dots, N$, where $\lambda_1 = (3 - \sqrt{3})/6$ and $\lambda_2 = (3 + \sqrt{3})/6$. Let $w(x) : \overline{\Omega} \to \overline{\Omega}$ be a bijective function in C^3 with $w'(x) > 0$ for all x and let w_i be the set of collocation points.

Theorem 2 *([10]). Under the assumptions of Theorem 1, and the assumption that $u'' - u^{3''}_{\Delta[1]}$ has a smooth expansion at the collocation points $u^3_\Delta \in S^3_{\Delta_w}$ defined by (2.28) exists, is unique, and satisfies the global error estimates*

$$\left\| \left(u - u^3_{\Delta}\right)^{(k)} \right\|_\infty = \mathcal{O}\left(h^{4-k}\right), \quad k = 0, 1, 2, 3,$$

and the local error estimates

$$\begin{aligned} \left| \left(u - u^3_\Delta\right)' (x) \right| &= \mathcal{O}\left(h^4\right) \quad \text{for } x = s_i \text{ and } w_i, \\ \left| \left(u - u^3_\Delta\right)'' (\sigma_{ij}) \right| &= \mathcal{O}\left(h^3\right), \\ \left| \left(u - u^3_\Delta\right)''' (w_i) \right| &= \mathcal{O}\left(h^2\right). \end{aligned}$$

The coefficient of y'' in Eq. (1.1), which is the constant $-\epsilon$, as well as p, q and f are assumed to be continuous; hence assumption (A1) of Theorem 1 is satisfied. We assume additional conditions imposed on the coefficients (see [1]) to ensure the existence of a unique solution of problem (1.1)–(1.2) and hence assumption (A2) is automatically valid. Since from (1.2) we have $y(a) = \alpha$ and $y(b) = \beta$, thus the solution of $y'' = 0$ subject to these two conditions is unique and is given by $y = \frac{\beta-\alpha}{b-a}(x-a)+\alpha$. Thus assumption (A3) of Theorem 2 is valid. We chose a number of grading functions $w := [a, b] \to [a, b]$ in (2.9)–(2.11). The first one is

$$w(x) = (b-a)\left[1 - \frac{(1+k)^{\left(1-\frac{x-a}{b-a}\right)} - 1}{k}\right] + a,$$

Table 2
Numerical solution of Example 1.

h	ϵ	Maximum error (our method)	Maximum error (method in [3])
1/32	0.1	3.0808(−5)	3.3(−2)
1/32	0.01	2.9509(−3)	2.9(−2)
1/32	0.001	1.3438(−3)	–
1/128	0.1	1.2122(−7)	9.4(−3)
1/128	0.01	2.1678(−5)	7.3(−3)
1/128	0.001	2.3552(−6)	–

Table 3
Rate of convergence of numerical solutions of Example 1 for $\epsilon = 0.01$.

h	1/16	1/32	1/64	1/128	1/256
Max error	1.732(−3)	7.530(−5)	4.811(−6)	3.206(−7)	2.172(−8)
Order		4.5235	3.9683	3.9072	3.8839

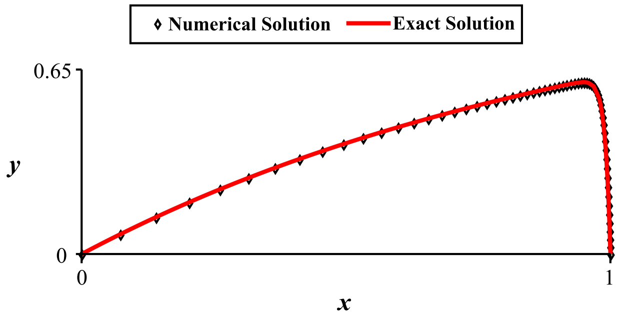

Fig. 1. Numerical solution of Example 1 for $\epsilon = 0.01$ and $h = 1/64$.

which is clearly a bijective map in $\mathbf{C}^3$ that takes the interval $[a, b]$ onto $[a, b]$. In addition, we have

$$w'(x) = \left[\frac{(1+k)^{\left(1-\frac{x-a}{b-a}\right)}}{k}\right] \ln(1+k),$$

which is clearly greater than zero for our choice $k > 0$. Moreover, we can easily obtain the inverse function, namely,

$$w^{-1}(x) = b - \frac{(b-a)}{\ln(1+k)} \ln\left[1 + k - \frac{k(x-a)}{b-a}\right].$$

Clearly $w^{-1} \in \mathbf{C}^1[a, b]$ and we will assume that the solution of (1.1)–(1.2) is smooth enough so that $u \in \mathbf{C}^6[a, b]$. Thus the last assumption (A4) is satisfied and therefore the conclusion of Theorem 1 follows. Likewise for our second choice of Chebyshev–Gauss–Lobatto function

$$z_i = \frac{1}{2}\left(1 - \cos(i\pi/N)\right), \quad i = 0, 1, \ldots, N,$$

with transformation

$$w_i = (1-\tau)\left(3z_i^2 - 2z_i^3\right) + \tau z_i,$$

the same analysis follows. We have

$$w_i' = 6(1-\tau)z_i(1-z_i) + \tau.$$

Noting that $z_i \in [0, 1]$ and since $\tau \in [0, 1]$, it follows that $w_i' > 0$. The inverse function theorem implies that $w_i^{-1} \in \mathbf{C}^1[0, 1]$. The assumptions of Theorem 2 are same as Theorem 1 in addition to the assumption that $u'' - u_{\Delta[1]}^{3''}$ has a smooth expansion at the collocation points $u_\Delta^3 \in S_{\Delta_w}^3$, which is true for our case. Hence the conclusion of Theorem 2 follows as well. These two later theorems confirm that our approach has order four rate of convergence.

3. Numerical examples

In this section, the adaptive spline collocation approach over a non-uniform Shishkin-like mesh is carried out to solve special cases of the singularly perturbed boundary value problems (1.1)–(1.2) using a prepared program in Maple code. The resulting solutions are contrasted with exact solutions and other existing numerical solutions obtained in [3] (see Figs. 1–4 and Tables 2–7). For all examples, it is shown numerically that the approach has an optimum rate of convergence of order 4

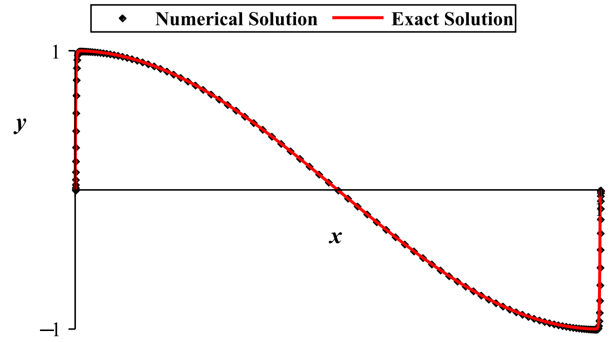

Fig. 2. Numerical solution of Example 2 for $p = 10^{-6}$ and $h = 1/128$.

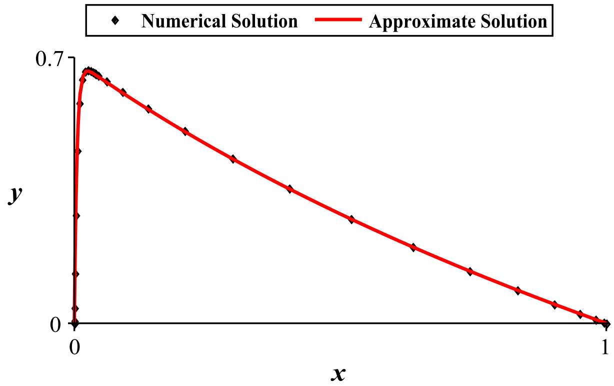

Fig. 3. Numerical solution of Example 3 for $\epsilon = 0.001$ and $h = 1/32$.

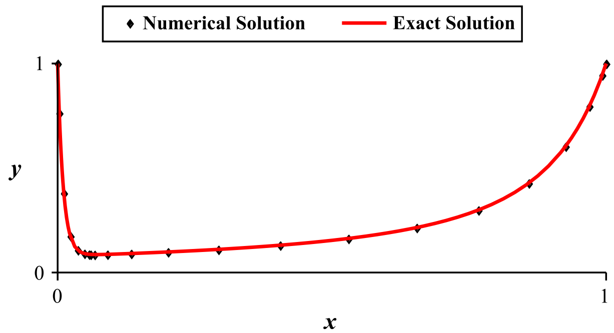

Fig. 4. Numerical solution of Example 4 for $\epsilon = 0.01$ and $h = 1/21$.

(see Tables 3 and 5–7) by utilizing the following log ratio formula:

$$p \approx \frac{\ln(\mathrm{Err}(n_i)) - \ln(\mathrm{Err}(n_{i+1}))}{\ln\left(\frac{b-a}{n_i}\right) - \ln\left(\frac{b-a}{n_{i+1}}\right)} \tag{3.29}$$

where

$$\mathrm{Err}(n) = \max_i \left|u_i^n - u(x_i, T)\right| .$$

Here u_i^n is the numerical value at (x_i, T) using n mesh intervals while $u(x_i, T)$ is the exact value at (x_i, T).

Example 1. Consider the convention-dominated equation:

$$-\epsilon y'' + y' + y = 1, \qquad y(0) = y(1) = 0.$$

The exact solution is given by: $y(x) = \frac{e^{\lambda_1 x}\left(e^{\lambda_2}-1\right)}{e^{\lambda_1}-e^{\lambda_2}} - \frac{e^{\lambda_2 x}\left(e^{\lambda_1}-1\right)}{e^{\lambda_1}-e^{\lambda_2}} + 1$, where $\lambda_1 = \frac{1+\sqrt{1+4\epsilon}}{2\epsilon}$ and $\lambda_1 = \frac{1-\sqrt{1+4\epsilon}}{2\epsilon}$.

Table 4
Numerical solution of Example 2 for $p = 10^{-6}$.

τ	h	ϵ	Maximum error (our method)	h	ϵ	Maximum error (method in [3])
0.05	1/128	10^{-2}	5.8309(−8)	1/1024	0.01	3.0(−3)
0.01	1/128	10^{-3}	4.9840(−7)	1/1024	0.0015	1.2(−3)
0.001	1/128	10^{-6}	2.5926(−5)			

Table 5
Rate of convergence of numerical solutions of Example 2 for $p = 10^{-6}$ and $\epsilon = 0.01$.

h	1/16	1/32	1/64	1/128	1/256
Max error	3.4073(−4)	2.0404(−5)	1.4916(−6)	9.5283(−8)	5.9618(−9)
Order		4.0617	3.7739	3.9685	3.9984

Table 6
Rate of convergence of numerical solutions of Example 3 for $\epsilon = 0.01$.

h	1/16	1/32	1/64	1/128
Error at $x = \sigma$	3.5685(−4)	1.8117(−5)	1.0171(−6)	5.8309(−8)
Order		4.2999	4.1548	4.1246

The layer's thickness is approximated by $\sigma = 0.083177$ and the uniformly distributed nodes are redistributed on $[0, 1-\sigma]$ and $[1-\sigma, 1]$ by using the grading function (2.10) and (2.9), respectively. The computational results are shown in Table 2 for different values of step size h and parameter ϵ, and the numerical solution for a special case is illustrated in Fig. 1. The order of the rate of convergence is presented in Table 3.

Example 2. Consider the following special case of the singularly perturbed boundary value problem (1.1)–(1.2):

$$-\epsilon y'' + py' + y = \cos(\pi x), \qquad y(0) = y(1) = 0,$$

where p is a constant. The exact solution is given by:

$$y(x) = A\cos(\pi x) + B\sin(\pi x) + Ce^{\lambda_1 x} + De^{-\lambda_2(1-x)},$$

where

$$A = \frac{1+\epsilon\pi^2}{p^2\pi^2 + (1+\epsilon\pi^2)^2}, \qquad B = \frac{p\pi}{p^2\pi^2 + (1+\epsilon\pi^2)^2}, \qquad C = -\frac{A(1+e^{-\lambda_2})}{1-e^{(\lambda_1-\lambda_2)}}, \qquad D = -\frac{A(1+e^{\lambda_1})}{1-e^{(\lambda_1-\lambda_2)}}.$$

Here $\lambda_1 < 0$ and $\lambda_2 > 0$ are the real solutions to $\epsilon\lambda^2 - p\lambda - 1 = 0$.

For this problem there exist boundary layers at both endpoints, so appropriately we chose the Lobatto function (2.11) with different values of the adjustment parameter τ depending on the magnitude of the parameter ϵ. The computational results are presented in Tables 4 and 5 and demonstrated in Fig. 2.

Example 3. Consider the nonlinear equation:

$$\epsilon y'' + 2y' = -e^y, \qquad y(0) = y(1) = 0.$$

Since no analytical solution exists for this problem we present the numerical solution at an approximate value of the location of the layer whose thickness is σ, namely at $x = 0.04$. The nodes are redistributed on $[0, \sigma]$ and $[\sigma, 1]$ according to the Lobatto function (2.11) with the choices $\tau = 0.1$ and $\tau = 0.5$, respectively. It is found that the error at this point is nearly maximum as compared with numerical solution at other node points.

Concerning the order of convergence, we considered the numerical solutions at $x = \sigma$ for different values of the step size h (see Table 6). As for the true solution at $x = \sigma$, we consider that for $h = 1/512$ (true solution equals 0.656870563668).

It is worth noting that in case the closed form solution is not available, the adaptive scheme upon using Newton's method given in (2.25) is iterated until the following condition is satisfied:

$$\sqrt{\sum_{i=0}^{N} |g^*(w_i, u_{n+1}) - g^*(w_i, u_n)|^2} < \epsilon$$

where ϵ is the required precision or tolerance. The number of iterations needed to satisfy this condition depends on the scheme, on the values of g^* and w_i.

Table 7
Maximum errors and rate of convergence of numerical solutions of Example 4 for $\epsilon = 0.01$.

h	1/21	1/42	1/84	1/168	1/336
Max error	1.5405(−3)	9.1499(−5)	5.6734(−6)	3.5403(−7)	2.1187(−8)
Order		4.0735	4.0115	4.0023	4.0005

Example 4. Consider the following linear equation that has a non-constant coefficient $p(x) = 1 - x$:

$$\epsilon y'' + (1-x)y' - y = 1, \qquad y(0) = y(1) = 1.$$

The layer's thickness is approximated by $\sigma = 0.06$ and the nodes are redistributed on $[0, \sigma]$ and $[\sigma, 1]$ according to the grading functions (2.10) and (2.9), respectively. The computational results and the order are presented in Table 7 and Fig. 4.

4. Conclusion

In this paper, an adaptive spline collocation method over a non-uniform Shishkin-like mesh is applied for the numerical solution of the singularly perturbed boundary value problem (1.1)–(1.2). Such problems exhibit a computational complexity due to the existence of a boundary layer. Despite that, the method is robust and handles the layer with high efficiency and accuracy. As is evident from the numerical results, this method gives $\mathcal{O}\left(h^4\right)$ convergence rate and the outcomes are better than the stated existing numerical techniques. The success in the applicability of the considered algorithm, for the stated problem, suggests that the same numerical method may be applied to the solution of other singular perturbed problems, with boundary or interior layers that have rough behavior, which might arise, for example, in the modeling of chemical reactions, theory of plasma and other fields.

Numerical examples are presented to illustrate the effectiveness of using a non-uniform mesh concentrated near the boundary layer. The non-uniform mesh is constructed in such a way that the mesh is finer and more dense near the boundary and the selection depends on the value of the perturbed parameter ϵ (which is finer for smaller ϵ). The non-uniform mesh produces more accurate approximations for the solution while using less nodes than required on a uniform mesh. In addition, when a non-uniform mesh is used with nodes that are condensed in the neighborhood of the boundary layers, the error will be uniformly distributed; however the rate of convergence stays the same.

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